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DERIVATION OF THE UNITED STATES MORTALITY TABLE BY OSCULATORY INTERPOLATION.

REGISTRATION STATES, TWELFTH CENSUS.

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INTRODUCTION.

The object of this paper is to derive a representative or standard mortality table from the best available population and vital statistics furnished by the United States Census Bureau. With this end in view I have made a study of the statistics for (1) aggregate males, and (2) native white, native parents, males. This paper deals with the first group, and the data employed are confined to the following eleven registration states: Connecticut, District of Columbia, Indiana, Maine, Massachusetts, Michigan, New Hampshire, New Jersey, New York, Rhode Island, and Vermont. The following statements, quoted from the official census reports of recent date, serve to define the term *registration states.*

"These are (the registration states) all states . . . for which the registration of deaths under local laws and ordinances was found to be sufficiently accurate for use by the Census Office." Abstract of the Twelfth Census, p. 178.

"For the decennial mortality statistics of the Twelfth Census, the 'registration area' was fixed after the records of deaths obtained from registration sources had been compared with the mortality returns of the census enumerators." Mortality Statistics, 1900–1904, p. xiv.

"The 'registration area'—that is to say, the states having

laws the results of whose operation have been accepted as giving practically complete mortality returns, . . ." Mortality Statistics, 1905, p. 5.

" . . . the completeness of registration of deaths for the state as a whole was equal to that of the minimum accepted for the registration area (90 per cent). . . ." Mortality Statistics, 1906, p. 27.

" . . . it is not yet certainly known whether any registration area, and especially any state area, in the United States has succeeded in obtaining an actual registration of at least 90 per cent of all births that occur — the minimum standard for the admission of a state or city to the registration area for deaths." Mortality Statistics, 1906, p. 16.

"The *registration area* of the United States, which embraces those states and separate cities accepted as having approximately complete registration of deaths based upon the requirement of compulsory burial permits, . . ." Bulletin 104, Mortality Statistics, 1908, p. 7.

It may fairly be assumed from the preceding statements that the best available data, namely, that of the registration area, is believed to represent not less than 90 per cent of the actual deaths occurring within such area. For the eleven states above mentioned the following table exhibits in quinquennial age groups the population as of June 1, 1900, and the deaths during the calendar year 1900. The figures relate to the aggregate group of males.

TABLE I.—FUNDAMENTAL DATA, UNITED STATES CENSUS BUREAU.

AGGREGATE MALES, REGISTRATION STATES.

AGE GROUP.	POPULATION JUNE 1, 1900.	DEATHS, 1900.	AGE GROUP.	POPULATION JUNE 1, 1900.	DEATHS, 1900.
5–9	998,284	4,688	55–59	341,797	8,475
10–14	911,960	2,646	60–64	275,523	9,333
15–19	881,784	4,286	65–69	201,124	10,125
20–24	915,560	6,412	70–74	140,705	10,221
25–29	911,834	7,171	75–79	85,026	9,348
30–34	831,959	7,258	80–84	40,356	6,791
35–39	756,051	7,671	85–89	13,704	3,378
40–44	645,445	7,482	90–94	2,997	1,037
45–49	517,413	7,230	95–99	551	220
50–54	444,151	7,934	100 and over	130	54

The population data were drawn from Table 2, pp. 18-99, Twelfth Census, Vol. II, the totals being found by adding together for each quinquennial age group the corresponding aggregate male population of the eleven registration states. The deaths, in quinquennial age groups, were taken from Table 4, pp. 60 61, line 52, Mortality Statistics, 1900–1904. From these data the final mortality table was derived by the method of osculatory interpolation in the manner hereafter explained.

THEORY OF OSCULATORY INTERPOLATION.

I next pass to the consideration of the formulæ employed in the reduction of the above data. One of the chief difficulties met with in adjusting rough data by the usual interpolation formulæ is that at the points where the interpolation curves meet there are sudden breaks in the values of the first differences which do not accord with our preconceived notion of the continuity of the function under discussion. Various methods have been resorted to in joining these interpolation curves; probably the most satisfactory, so far as results are concerned, is the *welding* process by means of sine curve factors employed in constructing the Registrar-General's English [1] Life Table, No. 6. The method employed in this paper, however, proceeds along quite different lines. The fundamental idea is to join two successive interpolation curves in such manner that for a certain abscissa they shall have a common ordinate, gradient, and radius of curvature. This is accomplished by making their first and second derivatives equal, respectively, at the point of intersection determined by the abscissa and common ordinate. In the case of interpolation curves of higher degree it will appear that the third and higher derivatives may be equal. This method has been termed "Osculatory Interpolation" because the successive interpolation curves have a common osculating circle at their points of junction; it was first devised by Sprague,[2] and later developed and applied by Karup,[3] King,[4] Buchanan,[5] and

[1] For description of method see Supplement, Part I, p. xvii, 65th Annual Report of the Registrar-General.

[2] Thomas Bond Sprague, Explanation of a New Formula for Interpolation, Jour. Inst. Actuaries, Vol. 22, p. 270.

[3] Johannes Karup, On a New Mechanical Method of Graduation, Trans. Second International Actuarial Congress, p. 78.

[4] George King, On the Construction of Mortality Tables from Census Returns and Records of Deaths, Jour. Inst. Actuaries, Vol. 42, p. 225.

[5] James Buchanan, Osculatory Interpolation by Central Differences; with an Application to Life Table Construction, Jour. Inst. Actuaries, Vol. 42, p. 369.

others. In developing the subject of osculatory formulæ I have not followed any of the above writers, but have generalized the osculatory formula along a line suggested by Lidstone.[1]

The problem is to find an interpolation curve y_x which shall be a blend of two curves u_x and v_x, so that

$$(1) \qquad y_x = (1 - c_x)u_x + c_x v_x,$$

where c_x is a rational algebraic function of x, and u_x and v_x are defined as follows:

$$(2) \quad u_x = u_{-n+(x+n)} = u_{-n} + {}_{x+n}C_1 \Delta u_{-n} + {}_{x+n}C_2 \Delta^2 u_{-n} + \cdots + {}_{x+n}C_{2n} \Delta^{2n} u_{-n},$$

$$(3) \quad v_x = v_{-(n-1)+(x+n-1)} = u_{-(n-1)} + {}_{x+n-1}C_1 \Delta u_{-(n-1)} + {}_{x+n-1}C_2 \Delta^2 u_{-(n-1)} + \cdots + {}_{x+n-1}C_{2n} \Delta^{2n} u_{-(n-1)},$$

where
$$ {}_mC_r = \frac{m(m-1)\cdots(m-r+1)}{1 \cdot 2 \cdots r}.$$

But
$$u_{k+1} = u_k + \Delta u_k,$$
and hence
$$c\Delta^s u_{k+1} = c\Delta^s u_k + c\Delta^{s+1} u_k.$$
Also
$${}_{x+n-1}C_{r-1} + {}_{x+n-1}C_r = {}_{x+n}C_r,$$
and therefore

$$(4) \quad v_x = u_{-n} + {}_{x+n}C_1 \Delta u_{-n} + \cdots + {}_{x+n}C_{2n} \Delta^{2n} u_{-n} + {}_{x+n-1}C_{2n} \Delta^{2n+1} u_{-n},$$

whence

$$(5) \qquad v_x = u_x + {}_{x+n-1}C_{2n} \Delta^{2n+1} u_{-n}.$$

It thus appears that u_x and v_x are rational, integral functions of x of degree $2n$ which pass through the ends of the first and last $2n+1$ ordinates, $u_{-n}, u_{-(n-1)}, \cdots u_{-1}, u_0, u_1, \cdots u_{n-1}, u_n, u_{n+1}$, respectively. They thus intersect or have in common the $2n$ points (k, u_k), where $k = -(n-1), \cdots -1, 0, 1, \cdots n-1, n$. Since by (1) the curve y_x passes through any point common to u_x and v_x, it must also pass through the above $2n$ points.

Let

$$(6) \qquad y_x = u_x + \psi_x,$$

where ψ_x is a rational integral function of x of degree not less than $2n+1$. We have also by (5) the relation

$$(7) \qquad v_x - u_x = {}_{x+n-1}C_{2n} \cdot \Delta^{2n+1} u_{-n}.$$

At the origin, $x = 0$, we assume that y_x and u_x, together with their first n derivatives, are equal, respectively, also at the point

[1] George J. Lidstone, Alternative Demonstration of the Formula for Osculatory Interpolation, Jour. Inst. Actuaries, Vol. 42, p. 394.

$x=1$, that y_x and v_x, together with their first n derivatives, are equal, respectively. Hence

$$(8)\qquad \begin{aligned} y_0 &= u_0,\\ y_0' &= u_0',\\ y_0'' &= u_0'',\\ &\cdots\\ y_0^{(n)} &= u_0^{(n)}. \end{aligned} \qquad\qquad (9)\qquad \begin{aligned} y_1 &= v_1,\\ y_1' &= v_1',\\ y_1'' &= v_1'',\\ &\cdots\\ y_1^{(n)} &= v_1^{(n)}. \end{aligned}$$

The relations (6) and (8) show that $\psi_0=\psi_0'=\psi_0''=\cdots=\psi_0^{(n)}=0$, hence ψ_x contains the factor x^{n+1}, while (5), (6), and the first one in (9) show that $\psi_1=0$, hence ψ_x contains the factor $x-1$. The remaining equations in (9) lead to the following set of relations:

$$(10)\qquad \begin{aligned} \psi_1' &= ({}_{x+n-1}C_{2n}')_{x=1}\cdot\Delta^{2n+1}u_{-n},\\ \psi_1'' &= ({}_{x+n-1}C_{2n}'')_{x=1}\cdot\Delta^{2n+1}u_{-n},\\ &\cdots\cdots\\ \psi_1^{(n)} &= ({}_{x+n-1}C_{2n}^{(n)})_{x=1}\cdot\Delta^{2n+1}u_{-n}. \end{aligned}$$

But

$$(11)\qquad \begin{aligned} \psi_x &= x^{n+1}(x-1)\phi_x\,\Delta^{2n+1}u_{-n}/\underline{|2n}\\ &=(x-1)\,\xi_x\,\Delta^{2n+1}u_{-n}/\underline{|2n}, \end{aligned}$$

where ϕ_x is a rational integral function of x of degree not less than $n-1$. If ϕ_x is taken of degree $n-1$, we have

$$(12)\qquad \phi_x=a_0x^{n-1}+a_1x^{n-2}+\cdots+a_{n-1},$$

$$(13)\qquad \xi_x=a_0x^{2n}+a_1x^{2n-1}+\cdots+a_{n-1}x^{n+1}.$$

Let $\psi_x=\omega_x\,\xi_x\,\Delta^{2n+1}u_{-n}/\underline{|2n}$,

where $\omega_x=x-1,\qquad \omega_x'=1,\qquad \omega_x''=\omega_x'''=\cdots=0,$

whence $\omega_1=0,\qquad \omega_1'=1,\qquad \omega_1''=\omega_1'''=\cdots=0.$

Then

$$\begin{aligned} \psi_x' &= \omega_x\xi_x'+\omega_x'\xi_x,\\ \psi_x'' &= \omega_x\xi_x''+2\,\omega_x'\xi_x'+\omega_x''\xi_x,\\ &\cdots\cdots\\ \psi_x^{(n)} &= \omega_x\xi_x^{(n)}+n\omega_x'\xi_x^{(n-1)}+c_2\omega_x''\xi_x^{(n-2)}+\cdots+c_n\omega_x^{(n)}\xi_x, \end{aligned}$$

where $c_2, c_3, \cdots c_n$, are constants. Therefore

$$(14)\qquad \psi_1^{(k)}=k\xi_1^{(k-1)}\cdot\Delta^{2n+1}u_{-n}/\underline{|2n}.$$

Similarly, since

$$\begin{aligned} &{}_{x+n-1}C_{2n}\Delta^{2n+1}u_{-n}\\ &=(x-1)[(x+n-1)\cdots x(x-2)\cdots(x-n)]\Delta^{2n+1}u_{-n}/\underline{|2n}\\ &=\omega_x\,\mu_x\,\Delta^{2n+1}u_{-n}/\underline{|2n}, \end{aligned}$$

where

$$(15)\qquad \mu_x = (x+n-1)\cdots x(x-2)\cdots(x-n),$$

we find that

$$(16)\qquad ({}_{x+n-1}C_{2n}{}^{(k)})_{x=1}\cdot\Delta^{2n+1}u_{-n} = k\mu_1{}^{(k-1)}\Delta^{2n+1}u_{-n}/\underline{|2\,n},$$

and comparing equations (10), (14), and (16), we have

$$(17)\qquad \xi_1{}^{(k-1)} = \mu_1{}^{(k-1)}\quad k=1,\,2,\,\cdots\,n,$$

which furnish the n linear equations determining the n constants $a_0, a_1, \cdots a_{n-1}$, of ϕ_x.

Having thus determined ψ_x we can find c_x by the relation

$$(18)\qquad \psi_x = c_x(v_x - u_x),$$

which taken with (7) gives

$$\psi_x = c_x\cdot{}_{x+n-1}C_{2n}\Delta^{2n+1}u_{-n},$$

whence

$$c_x = \frac{\psi_x}{{}_{x+n-1}C_{2n}\Delta^{2n+1}u_{-n}} = \frac{x^{n+1}\phi_x}{\mu_x},$$

and therefore we have

$$(19)\qquad c_x = \frac{x^n(a_0x^{n-1}+a_1x^{n-2}+\cdots+a_{n-1})}{(x+n-1)\cdots(x+1)(x-2)\cdots(x-n)}.$$

SPECIAL OSCULATORY FORMULÆ.

We shall now apply the preceding theory to obtain the osculatory formulæ in a few special cases. The simplest case is where $n=1$. Here we have four ordinates y_{-1}, y_0, y_1, y_2, and u_x is defined by the first three and v_x by the last three. From equations (6) and (11) we have

$$(20)\qquad y_x = u_x + x^2(x-1)\,a_0\Delta^3y_{-1}/\underline{|2},$$

where
$$u_x = y_{-1} + (x+1)\Delta y_{-1} + \frac{(x+1)x}{\underline{|2}}\cdot\Delta^2y_{-1},$$

and (17) furnishes the relation $a_0=1$, and (19) makes $c_x = x$, whence from (1) we have

$$y_x = (1-x)u_x + xv_x.$$

The expression for y_x given in (20) easily transforms into the following formula obtained by Karup:[1]

$$(21)\qquad y_x = y_0 + x\Delta y_0 + \frac{x(x-1)}{\underline{|2}}\Delta^2y_{-1} + \frac{x^2(x-1)}{\underline{|2}}\Delta^3y_{-1}.$$

[1] Karup, *loc. cit.*, p. 83.

When $n=2$, we have six ordinates, $y_{-2}, y_{-1}, y_0, y_1, y_2, y_3$, and u_x is defined by the first five and v_x by the last five. Equations (6) and (11) make

$$(22) \qquad y_x = u_x + x^3(x-1)(a_0x+a_1)\Delta^5 y_{-2}/\underline{|4},$$

where $u_x = y_{-2} + (x+2)\Delta y_{-2} + \frac{(x+2)(x+1)}{\underline{|2}}\Delta^2 y_{-2}$

$$+\frac{(x+2)(x+1)x}{\underline{|3}}\Delta^3 y_{-2} + \frac{(x+2)(x+1)x(x-1)}{\underline{|4}}\Delta^4 y_{-2}.$$

From equations (13) and (15) and (17)

$$\xi_x = a_0x^4 + a_1x^3, \qquad \mu_x = (x+1)\,x(x-2),$$

$$\xi_x' = 4\,a_0x^3 + 3\,a_1x^2, \quad \mu_x' = (x+1)x + x(x-2) + (x+1)(x-2).$$

$$\begin{aligned} a_0 + a_1 &= -2, \\ 4\,a_0 + 3\,a_1 &= -1. \end{aligned} \qquad \therefore\ a_0 = 5,\ a_1 = -7.$$

From (19) we have $c_x = \dfrac{x^2(5\,x-7)}{(x+1)(x-2)}$.

The above expression (22) for y_x readily transforms into the following formula arrived at by Karup:[1]

$$(23) \qquad y_x = y_0 + x\Delta y_0 + \frac{x(x-1)}{\underline{|2}}\Delta^2 y_{-1} + \frac{(x+1)x(x-1)}{\underline{|3}}\Delta^3 y_{-1}$$

$$+\frac{(x+1)x(x-1)(x-2)}{\underline{|4}}\Delta^4 y_{-2} + \frac{x^3(x-1)(5\,x-7)}{\underline{|4}}\Delta^5 y_{-2}.$$

When $n=3$ we have eight ordinates, $y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, y_4$, and u_x is determined by the first seven and v_x by the last seven. Formulæ (6) and (11) make

$$(24) \qquad y_x = u_x + x^4(x-1)(a_0x^2 + a_1x + a_2)\Delta^7 y_{-3}/\underline{|6},$$

where

$$u_x = y_{-3} + (x+3)\Delta y_{-3} + \frac{(x+3)(x+2)}{\underline{|2}}\Delta^2 y_{-3}$$

$$+\frac{(x+3)(x+2)(x+1)}{\underline{|3}}\Delta^3 y_{-3} + \frac{(x+3)(x+2)(x+1)x}{\underline{|4}}\Delta^4 y_{-3}$$

$$+\frac{(x+3)(x+2)(x+1)x(x-1)}{\underline{|5}}\Delta^5 y_{-3}$$

$$+\frac{(x+3)(x+2)(x+1)x(x-1)(x-2)}{\underline{|6}}\Delta^6 y_{-3}.$$

[1] Karup, *loc. cit.*, p. 83.

From relations (13), (15), and (17)

$$\xi_x = a_0x^6 + a_1x^5 + a_2x^4,$$
$$\xi_x' = 6\,a_0x^5 + 5\,a_1x^4 + 4\,a_2x^3,$$
$$\xi_x'' = 30\,a_0x^4 + 20\,a_1x^3 + 12\,a_2x^2,$$
$$\mu_x = (x+2)(x+1)x(x-2)(x-3)$$
$$= x^5 - 2\,x^4 - 7\,x^3 + 8\,x^2 + 12\,x,$$
$$\mu_x' = 5\,x^4 - 8\,x^3 - 21\,x^2 + 16\,x + 12,$$
$$\mu_x'' = 20\,x^3 - 24\,x^2 - 42\,x + 16,$$

whence

$$a_0 + a_1 + a_2 = 12,$$
$$6\,a_0 + 5\,a_1 + 4\,a_2 = 4,$$
$$30\,a_0 + 20\,a_1 + 12\,a_2 = -30,$$

and

$$a_0 = 89,\ \ a_1 = -222,\ \ a_2 = 145.$$

The osculatory interpolation curve of the seventh degree thus becomes

$$(25) \qquad y_x = u_x + \frac{x^4(x-1)(89\,x^2 - 22\,x + 145)}{\underline{|6}}\Delta^7 y_{-3},$$

and by (19) it may be put in the form

$$y_x = (1 - c_x)u_x + c_xv_x,$$

where

$$c_x = \frac{x^3(89\,x^2 - 222\,x + 145)}{(x+2)(x+1)(x-2)(x-3)}.$$

The above expression for y_x may easily be transformed into the following:

$$(26) \quad y_x = y_0 + x\Delta y_0 + \frac{x(x-1)}{\underline{|2}}\Delta^2 y_{-1} + \frac{(x+1)x(x-1)}{\underline{|3}}\Delta^3 y_{-1}$$
$$+ \frac{(x+1)x(x-1)(x-2)}{\underline{|4}}\Delta^4 y_{-2} + \frac{(x+2)(x+1)x(x-1)(x-2)}{\underline{|5}}\Delta^5 y_{-2}$$
$$+ \frac{(x+2)(x+1)x(x-1)(x-2)(x-3)}{\underline{|6}}\Delta^6 y_{-3}$$
$$+ \frac{x^4(x-1)(89\,x^2 - 222\,x + 145)}{\underline{|6}}\Delta^7 y_{-3}.$$

An examination of the cases for $n = 1, 2, 3$, makes it easy to write down the general formula. A very interesting characteristic of the osculatory formula is that it differs from the corresponding ordinary interpolation formula only in the last term.

This suggests that by properly modifying or correcting the highest differences we might mechanically perform the osculatory interpolation; the details of this method were worked out by Karup in his paper[1] on mechanical graduation. Another point worth emphasizing is that the osculatory interpolation curve may be regarded as a blend or combination of the two curves u_x and v_x in the proportion of $1-c_x$ to c_x. The function c_x vanishes at $x=0$, and $1-c_x$ vanishes at $x=1$; it is continuous in the interval from 0 to 1 and may be regarded as the welding factor employed in this interval to weld together the two curves u_x and v_x to form the curve y_x. While in this discussion equal unit intervals have been assumed between the ordinates, it is not essential that the intervals be equal. And the theory admits of easy generalization in such manner that u_x and v_x may differ in degree, and osculatory interpolation curves of different degree may follow one another in succession. In short, many other conditions besides those peculiar to osculatory interpolation curves may be imposed and satisfied by properly choosing the constants a_0, a_1, a_2, $\cdots$ both as to number and value.[2]

In order to interpolate *four* values between $x=0$ and $x=1$, we form the leading differences in the usual manner and arrive at the following relations, to be employed with the osculatory formula where $n=2$.

$$
(27)\quad
\begin{aligned}
\delta y_0 &= +.2\,\Delta y_{-2} + .32\,\Delta^2 y_{-2} + .088\,\Delta^3 y_{-2} - .0176\,\Delta^4 y_{-2} + .0016\,\Delta^5 y_{-2}\\
\delta^2 y_0 &= +.04\,\Delta^2 y_{-2} + .048\,\Delta^3 y_{-2} + .0016\,\Delta^4 y_{-2} + .0048\,\Delta^5 y_{-2}\\
\delta^3 y_0 &= +.008\,\Delta^3 y_{-2} + .0064\,\Delta^4 y_{-2} - .0048\,\Delta^5 y_{-2}\\
\delta^4 y_0 &= +.0016\,\Delta^4 y_{-2} - .0032\,\Delta^5 y_{-2}\\
\delta^5 y_0 &= +.0080\,\Delta^5 y_{-2}
\end{aligned}
$$

APPLICATION TO CENSUS DATA.

We are now ready to apply the osculatory method to the data of Table I. The formula employed is (22). Since the population and deaths are given for quinquennial age groups they may be considered as being represented graphically by areas. For this reason the areas are summed from the highest age back to age x and a new table of the functions T_x and l_x is formed and these values become the subject of interpolation. T_x then denotes the population aged x and over, l_x the number of deaths within the calendar year of persons aged x and over.

[1] Karup, *loc. cit.*, p. 84. [2] Lidstone, *loc. cit.*, p. 284.

TABLE II.—POPULATION AND DEATHS OF PERSONS AGED x AND OVER.

AGGREGATE MALES, REGISTRATION STATES.

AGE.	POPULATION JUNE 1, 1900.	DEATHS, 1900.	AGE.	POPULATION JUNE 1, 1900.	DEATHS, 1900.
x	T_x	l_x	x	T_x	l_x
5	8,916,354	121,760	55	1,101,913	58,982
10	7,918,070	117,072	60	760,116	50,507
15	7,006,110	114,426	65	484,593	41,174
20	6,124,326	110,140	70	283,469	31,049
25	5,208,766	103,728	75	142,764	20,828
30	4,296,932	96,557	80	57,738	11,480
35	3,464,973	89,299	85	17,382	4,689
40	2,708,922	81,628	90	3,678	1,311
45	2,063,477	74,146	95	681	274
50	1,546,064	66,916	100	130	54

The effect of interpolating T_x and l_x by the osculatory method is to reproduce the values given in Table II and thus leave the totals in each quinquennial age group in Table I unchanged. It thus happens that the irregularities in the original data due to crowding at the ages which are multiples of five are eliminated and a smooth curve obtained without altering the quinquennial aggregates. This, of course, is precisely what is desired. There are various methods of graduating the data given in Table II, among which I have chosen that denoted by King[1] as Construction A. This consists in finding by interpolation of T_x and l_x the values of L_x and d_x for each age. The latter lead directly by calculation to the value of the death rate q_x.

To apply the fifth difference osculatory interpolation formula (22) to T_x the differences are formed as shown in Table III. These *quinquennial* differences are denoted by Δ, Δ^2, etc. The leading *unit* differences, denoted by δ, δ^2, etc., are then found by employing the multipliers at the foot of Table III in connection with the quinquennial differences in this same table in accordance with the relations (27). Table IV exhibits these leading unit differences.

[1] King, *loc. cit.*, p. 253.

TABLE III.—QUINQUENNIAL DIFFERENCES OF POPULATION AGED x AND OVER.

x	ΔT_x	$\Delta^2 T_x$	$\Delta^3 T_x$	$\Delta^4 T_x$	$\Delta^5 T_x$
5	−998,284	+ 86,324	−56,148	− 7,804	+109,258
10	−911,960	+ 30,176	−63,952	+101,454	− 62,807
15	−881,784	− 33,776	+37,502	+ 38,647	−118,763
20	−915,560	+ 3,726	+76,149	− 80,116	+118,781
25	−911,834	+ 79,875	− 3,967	+ 38,665	− 55,937
30	−831,959	+ 75,908	+34,698	− 17,272	− 54,924
35	−756,051	+110,606	+17,426	− 72,196	+156,058
40	−645,445	+128,032	−54,770	+ 83,862	−149,034
45	−517,413	+ 73,262	+29,092	− 65,172	+109,377
50	−444,151	+102,354	−36,080	+ 44,205	− 66,310
55	−341,797	+ 66,274	+ 8,125	− 22,105	+ 31,345
60	−275,523	+ 74,399	−13,980	+ 9,240	− 15,509
65	−201,124	+ 60,419	− 4,740	− 6,269	− 740
70	−140,705	+ 55,679	−11,009	− 7,009	+ 9,082
75	− 85,026	+ 44,670	−18,018	+ 2,073	+ 5,611
80	− 40,356	+ 26,652	−15,945	+ 7,684	
85	− 13,704	+ 10,707	− 8,261		
90	− 2,997	+ 2,446			
95	− 551				

Multipliers of above Δ's to form leading unit differences.

$\delta^5 T_{x+10}$					+.0080
$\delta^4 T_{x+10}$				+.0016	−.0032
$\delta^3 T_{x+10}$			+.0080	+.0064	−.0048
$\delta^2 T_{x+10}$		+.0400	+.0480	+.0016	+.0048
$\delta\ T_{x+10}$	+.2000	+.3200	+.0880	−.0176	+.0016

TABLE IV.—LEADING UNIT DIFFERENCES OF POPULATION AGED x AND OVER.

x	$\delta^5 T_x$	$\delta^4 T_x$	$\delta^3 T_x$	$\delta^2 T_x$	δT_x
15	+ 874.0640	−362.1120	−1023.5680	+1269.8080	−176661.9808
20	− 502.4560	+363.3088	+ 439.1632	−2001.8032	−180249.5376
25	− 950.1040	+441.8768	+1117.4192	− 59.1712	−184735.1520
30	+ 950.2480	−508.2848	− 473.6992	+4246.1552	−173618.4768
35	− 447.4960	+240.8624	+ 484.2176	+2797.9504	−157925.8992
40	− 439.3920	+148.1216	+ 430.6784	+4410.5536	−138831.7072
45	+1248.4640	−614.8992	−1071.7248	+5894.2528	−112762.4496
50	−1192.2720	+611.0880	+ 813.9200	+1911.1360	− 94652.9456
55	+ 875.0160	−454.2816	− 709.3744	+4747.6304	− 76156.6336
60	− 530.4800	+282.9200	+ 312.5600	+2114.7600	− 60136.0640
65	+ 250.7600	−135.6720	− 226.9280	+3156.0480	− 45997.5200
70	− 124.0720	+ 64.4128	+ 21.7392	+2245.2608	− 32714.5984
75	− 5.9200	− 7.6624	− 74.4896	+2175.6576	− 21198.6896
80	+ 72.6560	− 40.2768	− 176.5232	+1731.1072	− 11154.6224
85	+ 44.8880	− 14.6384	− 157.8096	+ 952.1856	− 4323.8912

By employing these leading unit differences we are now able to derive the T_x table for each age from 15 to 90. It is not necessary, however, to go beyond the δT_x column in this computation, as we desire only the population or L_x table for each age. The numerical check on the work appears in the summation of the five first differences in each interpolation group, for this sum should be equal to the difference between the corresponding pair of quinquennial values of T_x taken from Table III. This process is illustrated in the following calculation for the two age groups 15–19 and 35–39.

FIRST NUMERICAL ILLUSTRATION.

	Age Group 15–19.		Age Group 35–39.
	+874.0640		−447.4960
	−362.1120		+240.8624
	−1023.5680		+484.2176
	+1269.8080		+2797.9504
	−176661.9808		−157925.8992
	+511.9520		−206.6336
	−1385.6800		+725.0800
	+246.2400		+3282.1680
	−175392.1728		−155127.9488
	−873.7280		+518.4464
	−1139.4400		+4007.2480
	−175145.9328		−151845.7808
	−2013.1680		+ 4525.6944
	−176285.3728		−147838.5328
	−178298.5408		−143312.8384
15	176661.9808	35	157925.8992
16	175392.1728	36	155127.9488
17	175145.9328	37	151845.7808
18	176285.3728	38	147838.5328
19	178298.5408	39	143312.8384
15–19	$881784.0000 = T_{15} - T_{20}$	35–39	$756051.0000 = T_{35} - T_{40}$

Starting with the five leading unit differences opposite age 15 in Table IV, we form by successive addition in the usual manner, as appearing in the first column of the numerical illustration, the values of δT_x for the four ages between 15 and 20; a check on the correctness of the numerical work in this column and the calculation of the five leading unit differences from the five leading quinquennial differences *of an age ten years younger* is the fact that the sum, $-(\delta T_{15} + \delta T_{16} + \delta T_{17} + \delta T_{18} + \delta T_{19}) = 881{,}784$, agrees with the negative quinquennial difference, $-\Delta T_{15} = T_{15} - T_{20} = 881{,}784$, taken from Table III. This check, however, does not insure the correctness of the leading quinquennial differences employed in the calculation, for $\Delta^4 T_{x-10}$ and $\Delta^5 T_{x-10}$ in particular may be chosen at pleasure and the above check satisfied if the numerical work is correct. This is made apparent by an examination of the following numerical illustration, which exhibits the process of deriving the interpolated values by the successive addition of the leading unit differences, which in turn are expressed in terms of the leading quinquennial differences of an age ten years younger. The coefficients of the respective quinquennial differences appear in the columns below them.

SECOND NUMERICAL ILLUSTRATION.

	ΔT_{x-10}	$\Delta^2 T_{x-10}$	$\Delta^3 T_{x-10}$	$\Delta^4 T_{x-10}$	$\Delta^5 T_{x-10}$
$\delta^5 T_x$	.0000	.0000	.0000	.0000	+.0080
$\delta^4 T_x$	.0000	.0000	.0000	+.0016	−.0032
$\delta^3 T_x$	.0000	.0000	+.0080	+.0064	−.0048
$\delta^2 T_x$	.0000	+.0400	+.0480	+.0016	+.0048
δT_x	+.2000	+.3200	+.0880	−.0176	+.0016
	.0000	.0000	.0000	+.0016	+.0048
	.0000	.0000	+.0080	+.0080	−.0080
	.0000	+.0400	+.0560	+.0080	.0000
	+.2000	+.3600	+.1360	−.0160	+.0064
	.0000	.0000	+.0080	+.0096	−.0032
	.0000	+.0400	+.0640	+.0160	−.0080
	+.2000	+.4000	+.1920	−.0080	+.0064
	.0000	+.0400	+.0720	+.0256	−.0112
	+.2000	+.4400	+.2560	+.0080	−.0016
	+.2000	+.4800	+.3280	+.0336	−.0128
δT_x	+.2000	+.3200	+.0880	−.0176	+.0016
δT_{x+1}	+.2000	+.4800	+.3280	+.0336	−.0128
δT_{x+2}	+.2000	+.4400	+.2560	+.0080	−.0016
δT_{x+3}	+.2000	+.4000	+.1920	−.0080	+.0064
δT_{x+4}	+.2000	+.3600	+.1360	−.0160	+.0064
$\sum_{x}^{x+4} \delta T_x$	+1.0000	+2.0000	+1.0000	.0000	.0000

The final summation in the above numerical illustration, which corresponds to our so-called check in the first numerical illustration, appears to be merely a statement of the known relation

$$\sum_{x}^{x+4} \delta T_x = \Delta T_{x-10} + 2\,\Delta^2 T_{x-10} + \Delta^3 T_{x-10} = \Delta T_x,$$

which, indeed, we might have predicted in advance. Since the multipliers or coefficients of $\Delta^4 T_{x-10}$ and $\Delta^5 T_{x-10}$ combine in such manner in this process as to vanish, it is evident that these two differences may be taken arbitrarily, and the relation

$$\sum_{x}^{x+4} \delta T_x = \Delta T_x = T_{x+5} - T_x$$

will still be satisfied if the numerical work is correct.

The general relation would be as follows:

$$\Delta T_x = \Delta T_{x-5n} + n\Delta^2 T_{x-5n} + \frac{n(n-1)}{\underline{|2}} \Delta^3 T_{x-5n} + \cdots + \Delta^{n+1} T_{x-5n},$$

from which it appears that the last n quinquennial differences,

$$\Delta^{n+2}T_{x-5n}, \Delta^{n+3}T_{x-5n}, \cdots \Delta^{2n+1}T_{x-5n},$$

may be arbitrarily chosen in the general case, $n = n$, without disturbing the relation

$$\sum_{x}^{x+4} \delta T_x = \Delta T_x.$$

It is therefore evident that the differences in Table III must be first tested by some other formula, such as

$$\sum_{1}^{n} \Delta u_n = u_{n+1} - u_1,$$

and their correctness absolutely assured before we employ them to form the leading unit differences in Table IV and proceed to the calculation of the interpolated values of δT_x. It would be a useless labor and a fallacious test to add the respective columns of leading quinquennial differences in Table III and apply to these sums the multipliers at the foot of the table and check the same against the sums of the corresponding columns of the leading unit differences in Table IV. I have dwelt at some length on this point because King[1] offers it as "a complete final check on the whole work," and with the hope that this word of caution may be of assistance to others undertaking calculations of this nature.[2]

Returning to the examination of the first numerical illustration, we observe that the aggregate population in the quinquennial age group 15–19 remains undisturbed and the redistribution is made in such manner that the figures form the first differences, apart from the negative sign, of a function T_x which has been made the subject of an osculatory interpolation by fifth differences. The following table exhibits the enumerated and graduated population for each age from fifteen to eighty-nine, both inclusive.

[1] King, *loc. cit.*, p. 244.

[2] After this discussion was written I found that Sprague had noted the same thing in connection with the fifth difference equation, where $n = 3$. Sprague, *loc. cit.*, p. 284.

TABLE V.—POPULATION, ENUMERATED AND GRADUATED.

AGGREGATE MALES, REGISTRATION STATES.

AGE.	CENSUS, JUNE 1, 1900.	GRADUATED.	AGE.	CENSUS, JUNE 1, 1900.	GRADUATED.
15	177,666	176,662	45	131,636	112,762
16	180,668	175,392	46	96,579	106,868
17	174,772	175,146	47	94,312	102,046
18	176,597	176,285	48	100,247	98,910
19	172,081	178,299	49	94,639	96,827
15–19	881,784	881,784	45–49	517,413	517,413
20	178,264	180,250	50	126,236	94,653
21	180,837	182,251	51	75,903	92,742
22	182,552	183,814	52	87,851	90,017
23	183,814	184,574	53	76,617	85,866
24	190,093	184,671	54	77,544	80,873
20–24	915,560	915,560	50–54	444,151	444,151
25	193,570	184,735	55	85,830	76,157
26	180,066	184,794	56	72,184	71,409
27	181,020	183,736	57	62,508	67,371
28	189,633	181,119	58	62,175	64,496
29	167,545	177,450	59	59,100	62,364
25–29	911,834	911,834	55–59	341,797	341,797
30	215,807	173,619	60	79,876	60,136
31	141,323	169,372	61	45,214	58,021
32	167,874	165,600	62	52,610	55,594
33	154,650	162,809	63	50,989	52,571
34	152,305	160,559	64	46,834	49,201
30–34	831,959	831,959	60–64	275,523	275,523
35	176,535	157,926	65	52,700	45,998
36	143,012	155,128	66	40,688	42,841
37	135,897	151,846	67	39,672	39,912
38	157,661	147,838	68	35,659	37,346
39	142,946	143,313	69	32,405	35,027
35–39	756,051	756,051	65–69	201,124	201,124
40	181,577	138,832	70	40,103	32,715
41	108,908	134,421	71	25,519	30,469
42	132,531	129,580	72	28,131	28,202
43	112,420	124,160	73	24,757	25,849
44	110,009	118,452	74	22,195	23,470
40–44	645,445	645,445	70–74	140,705	140,705

TABLE V.—POPULATION, ENUMERATED AND GRADUATED.—*Continued.*

AGGREGATE MALES, REGISTRATION STATES.

AGE.	CENSUS, JUNE 1, 1900.	GRADUATED.	AGE.	CENSUS, JUNE 1, 1900.	GRADUATED.
75	23,496	21,199	85	4,420	4,324
76	18,599	19,023	86	3,068	3,372
77	15,852	16,922	87	2,675	2,577
78	14,640	14,903	88	2,052	1,955
79	12,439	12,979	89	1,489	1,476
75–79	85,026	85,026	85–89	13,704	13,704
80	12,680	11,155			
81	8,750	9,423			
82	7,514	7,869			
83	6,030	6,531			
84	5,382	5,378			
80–84	40,356	40,356			

It will be noticed that there is a strong tendency to mass about ages which are multiples of five, especially in the section of the table between ages 30 and 60. About age 30, for example, the adjustment amounts to a considerable change in the enumerated data, the corrections appearing as follows:—

AGE.	CENSUS.	GRADUATED.	CORRECTION.
28	189,633	181,119	− 8,514
29	167,545	177,450	+ 9,905
30	215,807	173,619	−42,188
31	141,323	169,372	+28,049
32	167,874	165,600	− 2,274

While we are impressed with the remarkable smoothness of the results obtained by the osculatory interpolation, we are compelled to regard these adjustments involving such extensive shifting of the figures as open to the objection of being artificial, perhaps in a high degree. Although we are certain that the enumerated population at age 30 is too great, it would be going too far to assert that this method of adjustment is the one most likely to give the approximately true figures.

We have next to deal with the vital statistics. The data given in the l_x column in Table II is now treated in the same manner as T_x in the preceding paragraphs. In fact, to T_x and L_x in that discussion, correspond l_x and d_x, respectively, in what follows.

Table VI gives the quinquennial differences of l_x, together with the multipliers to be used in forming the leading unit differences to be employed to obtain the interpolated values of d_x for each age. Table VII exhibits these calculated leading unit differences.

TABLE VI.—QUINQUENNIAL DIFFERENCES OF DEATHS AT AGE x AND OVER.

x	Δl_x	$\Delta^2 l_x$	$\Delta^3 l_x$	$\Delta^4 l_x$	$\Delta^5 l_x$
5	− 4,688	+2,042	−3,682	+3,196	−1,343
10	− 2,646	−1,640	− 486	+1,853	−2,548
15	− 4,286	−2,126	+1,367	− 695	− 303
20	− 6,412	− 759	+ 672	− 998	+1,926
25	− 7,171	− 87	− 326	+ 928	−1,467
30	− 7,258	− 413	+ 602	− 539	− 480
35	− 7,671	+ 189	+ 63	−1,019	+2,138
40	− 7,482	+ 252	− 956	+1,119	−1,599
45	− 7,230	− 704	+ 163	− 480	+ 863
50	− 7,934	− 541	− 317	+ 383	+ 247
55	− 8,475	− 858	+ 66	+ 630	− 357
60	− 9,333	− 792	+ 696	+ 273	+ 442
65	−10,125	− 96	+ 969	+ 715	−1,543
70	−10,221	+ 873	+1,684	− 828	−1,100
75	− 9,348	+2,557	+ 856	−1,928	+1,476
80	− 6,791	+3,413	−1,072	− 452	
85	− 3,378	+2,341	−1,524		
90	− 1,037	+ 817			
95	− 220				

Multipliers of above Δ's to form leading unit differences.

$\delta^5 l_{x+10}$					+.0080
$\delta^4 l_{x+10}$				+.0016	−.0032
$\delta^3 l_{x+10}$			+.0080	+.0064	−.0048
$\delta^2 l_{x+10}$		+.0400	+.0480	+.0016	+.0048
$\delta\, l_{x+10}$	+.2000	+.3200	+.0880	−.0176	+.0016

TABLE VII.—LEADING UNIT DIFFERENCES OF DEATHS AGED x AND OVER.

x	$\delta^5 l_x$	$\delta^4 l_x$	$\delta^3 l_x$	$\delta^2 l_x$	δl_x
15	−10.7440	+ 9.4112	− 2.5552	− 96.3888	− 666.5744
20	−20.3840	+11.1184	+20.2016	− 98.1936	−1133.4576
25	− 2.4240	− 0.1424	+ 7.9424	− 21.9904	−1405.4768
30	+15.4080	− 7.7600	−10.2560	+ 9.5440	−1445.4976
35	−11.7360	+ 6.1792	+10.3728	− 24.6848	−1509.4080
40	− 3.8400	+ 0.6736	+ 3.6704	+ 9.2096	−1522.0656
45	+17.1040	− 8.4720	−16.2800	+ 19.2160	−1446.8208
50	−12.7920	+ 6.9072	+ 7.1888	− 41.6928	−1522.1408
55	+ 6.9040	− 3.5296	− 5.9104	− 16.9616	−1647.1072
60	+ 1.9760	− 0.1776	− 1.2704	− 35.0576	−1794.1616
65	− 2.8560	+ 2.1504	+ 6.2736	− 31.8576	−1975.4112
70	+ 3.5360	− 0.9776	+ 5.1936	+ 4.2864	−2062.8896
75	−12.3440	+ 6.0816	+19.7344	+ 36.4096	−1985.5008
80	− 8.8000	+ 2.1952	+13.4528	+109.1472	−1603.8352
85	+11.8080	− 7.8080	−12.5760	+147.3680	− 939.7376

The third numerical illustration, which appears below, exhibits the process of calculating the values of d_x for the age groups 15–19 and 35–39. A check on the numerical work in the first column and the calculation of the five leading unit differences opposite age 15 in Table VII from the five leading quinquennial differences opposite age 5 in Table VI is obtained because $-(\delta l_{15} + \delta l_{16} + \delta l_{17} + \delta l_{18} + \delta l_{19}) = 4286$ agrees with the negative quinquennial difference, $-\Delta l_{15} = l_{15} - l_{20} = 4286$, taken from Table VI. This check does not insure the correctness of the leading quinquennial differences opposite age 5 in Table VI.

THIRD NUMERICAL ILLUSTRATION.

	Age Group 15–19.		Age Group 35–39.
	− 10.7440		− 11.7360
	+ 9.4112		+ 6.1792
	− 2.5552		+ 10.3728
	− 96.3888		− 24.6848
	−666.5744		−1509.4080
	− 1.3328		− 5.5568
	+ 6.8560		+ 16.5520
	− 98.9440		− 14.3120
	−762.9632		−1534.0928
	+ 5.5232		+ 10.9952
	− 92.0880		+ 2.2400
	−861.9072		−1548.4048
	− 86.5648		+ 13.2352
	−953.9952		−1546.1648
	−1040.5600		−1532.9296
15	666.5744	35	1509.4080
16	762.9632	36	1534.0928
17	861.9072	37	1548.4048
18	953.9952	38	1546.1648
19	1040.5600	39	1532.9296
15–19	$4286.0000 = l_{15} - l_{20}$	35–39	$7671.0000 = l_{35} - l_{40}$

The next table furnishes a comparison between the observed and graduated data with respect to deaths. It was made possible through the courtesy of Dr. Cressy L. Wilbur, Chief Statistician of the Division of Vital Statistics, who kindly furnished me with some of the advance sheets of the volume on Mortality Statistics for 1908. The reported deaths for the registration states in the calendar year 1900 are here given for each age, thus affording a comparison of deaths similar to that set forth on population in Table V. The process of osculatory interpolation having been applied to the quinquennial values of the function l_x, the deaths at each age, apart from the algebraic sign, appear as the first differences of this function, consequently the aggregates in the quinquennial age groups are unaltered in value but subjected to a new distribution of the individual ages within the age group. The general effect, as seen by an examination of Table VIII, is to smooth out the irregularities due to massing at the ages which are multiples of five. This table, together with Tables IX, X, and XI, appears at the end of the paper.

Table IX brings together the graduated population and deaths for each age, and the remaining column exhibits the corresponding death rates. The following formula was employed in computing the death rate: —

$$q_x = \frac{2\,d_x}{2\,L_x + d_x} = \frac{d_x}{L_x + \frac{1}{2}\,d_x}.$$

It is based upon the assumption that deaths occur uniformly throughout the year, so that at the beginning of the year there are $L_x + \frac{1}{2}\,d_x$ persons aged x, among whom d_x deaths occur at uniform intervals during the year, leaving a population of L_x in the middle and $L_x - \frac{1}{2}\,d_x$ at the end of the year. The death rate is given to six places of decimals.

Table X exhibits the death rate cut down to five places of decimals, and the third difference of the death rate multiplied by 100,000. This table is designed to indicate the extreme smoothness of the graduation of the death rate. It is possible that by further refinements some little improvement might be made in this particular, but I am inclined to think it would be unimportant and that the rates as they now stand closely represent the average actual conditions in the registration area of the United States.

Table XI is added to facilitate a comparison between the census death rates in this country and England. The New York City table, which was graduated by Roche[1] from census data, is exhibited to show the striking excess in the death rate in New York City over the average of the registration states. It is not my purpose, however, to discuss any of these features in this paper. The English Life Table, No. 6, was graduated by King.[2] A comparison shows that from ages 15 to 37 the death rate in this country is higher than in England, but after age 37 the reverse is true.

The death rates for ages preceding 15 and following 89 have not been considered in this paper, but I hope at no distant time to fill in this gap and construct the complete mortality table.

[1] John F. Roche, An Investigation into the Mortality Rates of the City of New York, Transactions Actuarial Society of America, Vol. 7, p. 435.

[2] King, *loc. cit.*, p. 274.

TABLE VIII.—COMPARISON OF REPORTED AND GRADUATED DEATHS.

AGGREGATE MALES, REGISTRATION STATES.

AGE.	REPORTED 1900.	GRADUATED.	AGE.	REPORTED 1900.	GRADUATED.	AGE.	REPORTED 1900.	GRADUATED.
15	603	667	40	1,976	1,522	65	2,292	1,975
16	754	763	41	1,194	1,513	66	1,830	2,007
17	847	862	42	1,588	1,500	67	2,004	2,033
18	976	954	43	1,394	1,483	68	2,063	2,050
19	1,106	1,040	44	1,330	1,464	69	1,936	2,060
15–19	4,286	4,286	40–44	7,482	7,482	65–69	10,125	10,125
20	1,108	1,133	45	1,748	1,447	70	2,436	2,063
21	1,249	1,232	46	1,284	1,428	71	1,728	2,059
22	1,290	1,310	47	1,256	1,425	72	2,066	2,049
23	1,306	1,356	48	1,487	1,446	73	1,987	2,035
24	1,459	1,381	49	1,455	1,484	74	2,004	2,015
20–24	6,412	6,412	45–49	7,230	7,230	70–74	10,221	10,221
25	1,415	1,405	50	1,985	1,522	75	2,191	1,986
26	1,398	1,427	51	1,258	1,564	76	1,983	1,949
27	1,439	1,442	52	1,605	1,598	77	1,721	1,893
28	1,518	1,448	53	1,496	1,619	78	1,859	1,811
29	1,401	1,449	54	1,590	1,631	79	1,594	1,709
25–29	7,171	7,171	50–54	7,934	7,934	75–79	9,348	9,348
30	1,702	1,445	55	1,903	1,647	80	1,813	1,604
31	1,174	1,436	56	1,754	1,664	81	1,341	1,495
32	1,584	1,437	57	1,626	1,687	82	1,356	1,372
33	1,449	1,455	58	1,685	1,719	83	1,146	1,234
34	1,349	1,485	59	1,507	1,758	84	1,135	1,086
30–34	7,258	7,258	55–59	8,475	8,475	80–84	6,791	6,791
35	1,755	1,509	60	2,237	1,794	85	951	940
36	1,410	1,534	61	1,503	1,829	86	766	792
37	1,362	1,549	62	1,791	1,866	87	663	658
38	1,732	1,546	63	1,868	1,903	88	550	543
39	1,412	1,533	64	1,934	1,941	89	448	445
35–39	7,671	7,671	60–64	9,333	9,333	85–89	3,378	3,378

TABLE IX.—ADJUSTED POPULATION, DEATHS, AND DEATH RATES.

AGGREGATE MALES, REGISTRATION STATES.

AGE.	POPULATION JUNE 1, 1900.	DEATHS, 1900.	DEATH RATE.	AGE.	POPULATION JUNE 1, 1900.	DEATHS, 1900.	DEATH RATE.
x	L_x	d_x	q_x	x	L_x	d_x	q_x
15	176,662	667	.003768	55	76,157	1,647	.021395
16	175,392	763	.004341	56	71,409	1,664	.023034
17	175,146	862	.004910	57	67,371	1,687	.024731
18	176,285	954	.005397	58	64,496	1,719	.026302
19	178,299	1,040	.005816	59	62,364	1,758	.027798
20	180,250	1,133	.006266	60	60,136	1,794	.029394
21	182,251	1,232	.006737	61	58,021	1,829	.031034
22	183,814	1,310	.007101	62	55,594	1,866	.033011
23	184,574	1,356	.007320	63	52,571	1,903	.035555
24	184,671	1,381	.007450	64	49,201	1,941	.038687
25	184,735	1,405	.007577	65	45,998	1,975	.042034
26	184,794	1,427	.007692	66	42,841	2,007	.045775
27	183,736	1,442	.007818	67	39,912	2,033	.049672
28	181,119	1,448	.007963	68	37,346	2,050	.053426
29	177,450	1,449	.008132	69	35,027	2,060	.057132
30	173,619	1,445	.008288	70	32,715	2,063	.061132
31	169,372	1,436	.008443	71	30,469	2,059	.065368
32	165,600	1,437	.008640	72	28,202	2,049	.070108
33	162,809	1,455	.008897	73	25,849	2,035	.075745
34	160,559	1,485	.009206	74	23,470	2,015	.082320
35	157,926	1,509	.009510	75	21,199	1,986	.089492
36	155,128	1,534	.009840	76	19,023	1,949	.097462
37	151,846	1,549	.010149	77	16,922	1,893	.105941
38	147,838	1,546	.010403	78	14,903	1,811	.114559
39	143,313	1,533	.010640	79	12,979	1,709	.123541
40	138,832	1,522	.010903	80	11,155	1,604	.134147
41	134,421	1,513	.011193	81	9,423	1,495	.146994
42	129,580	1,500	.011509	82	7,869	1,372	.160374
43	124,160	1,483	.011873	83	6,531	1,234	.172636
44	118,452	1,464	.012284	84	5,378	1,086	.183415
45	112,762	1,447	.012751	85	4,324	940	.196078
46	106,868	1,428	.013274	86	3,372	792	.210191
47	102,046	1,425	.013867	87	2,577	658	.226428
48	98,910	1,446	.014513	88	1,955	543	.243881
49	96,827	1,484	.015210	89	1,476	445	.261996
50	94,653	1,522	.015952				
51	92,742	1,564	.016723				
52	90,017	1,598	.017596				
53	85,866	1,619	.018679				
54	80,873	1,631	.019966				

TABLE X.—AMERICAN LIFE TABLE. AGGREGATE MALES.

REGISTRATION STATES. UNITED STATES CENSUS, 1900.

AGE.	q_x	$10^5\Delta^3 q_x$	AGE.	q_x	$10^5\Delta^3 q_x$
15	.00377	− 8	55	.02140	− 20
16	.00434	+ 1	56	.02303	+ 10
17	.00491	+10	57	.02473	+ 12
18	.00540	− 1	58	.02630	− 4
19	.00582	−13	59	.02780	+ 29
20	.00627	− 3	60	.02939	+ 23
21	.00674	+ 5	61	.03103	+ 1
22	.00710	+ 9	62	.03301	− 37
23	.00732	− 2	63	.03556	+ 20
24	.00745	+ 4	64	.03869	− 27
25	.00758	− 1	65	.04203	− 27
26	.00769	+ 2	66	.04578	+ 7
27	.00782	− 4	67	.04967	+ 36
28	.00796	0	68	.05343	− 6
29	.00813	+ 6	69	.05713	+ 26
30	.00829	+ 1	70	.06113	+ 40
31	.00844	− 1	71	.06537	+ 3
32	.00864	− 6	72	.07011	− 33
33	.00890	+ 4	73	.07575	+ 20
34	.00921	− 5	74	.08232	− 29
35	.00951	− 4	75	.08949	− 37
36	.00984	+ 5	76	.09746	+ 22
37	.01015	+ 3	77	.10594	+127
38	.01040	+ 1	78	.11456	+ 60
39	.01064	0	79	.12354	−169
40	.01090	+ 1	80	.13415	−165
41	.01119	+ 1	81	.14699	− 38
42	.01151	+ 1	82	.16037	+337
43	.01187	− 1	83	.17264	− 43
44	.01228	+ 3	84	.18342	+ 68
45	.01275	− 4	85	.19608	− 92
46	.01327	+ 2	86	.21019	− 54
47	.01387	− 2	87	.22643	
48	.01451	− 1	88	.24388	
49	.01521	+ 8	89	.26200	
50	.01595	+ 9			
51	.01672	+ 1			
52	.01760	− 7			
53	.01868	+ 6			
54	.01997	−19			

TABLE XI.—COMPARATIVE TABLE OF DEATH RATES, MALES.

Age.	Registration States.	New York City.	English Life Table No. 6.	Age.	Registration States.	New York City.	English Life Table No. 6.
x	q_x	q_x	q_x	x	q_x	q_x	q_x
15	.00377	.00392	.00303	55	.02140	.03284	.02558
16	.00434	.00430	.00322	56	.02303	.03477	.02726
17	.00491	.00465	.00351	57	.02473	.03676	.02917
18	.00540	.00510	.00387	58	.02630	.03926	.03127
19	.00582	.00581	.00425	59	.02780	.04203	.03351
20	.00627	.00664	.00459	60	.02939	.04421	.03583
21	.00674	.00751	.00487	61	.03103	.04714	.03823
22	.00710	.00814	.00508	62	.03301	.05034	.04069
23	.00732	.00864	.00522	63	.03556	.05460	.04328
24	.00745	.00908	.00537	64	.03869	.05926	.04617
25	.00758	.00966	.00555	65	.04203	.06328	.04949
26	.00769	.01019	.00575	66	.04578	.06584	.05328
27	.00782	.01058	.00598	67	.04967	.07075	.05765
28	.00796	.01102	.00624	68	.05343	.07569	.06251
29	.00813	.01152	.00653	69	.05713	.08068	.06763
30	.00829	.01203	.00685	70	.06113	.08547	.07279
31	.00844	.01240	.00720	71	.06537	.09067	.07801
32	.00864	.01293	.00761	72	.07011	.09488	.08319
33	.00890	.01332	.00805	73	.07575	.09939	.08854
34	.00921	.01387	.00853	74	.08232	.10558	.09450
35	.00951	.01442	.00903	75	.08949	.11236	.10145
36	.00984	.01491	.00955	76	.09746	.11765	.10948
37	.01015	.01545	.01009	77	.10594	.12639	.11878
38	.01040	.01631	.01065	78	.11456	.14159	.12943
39	.01064	.01699	.01122	79	.12354	.15385	.14136
40	.01090	.01736	.01179	80	.13415	.16561	.15457
41	.01119	.01787	.01236	81	.14699	.18040	.16923
42	.01151	.01843	.01291	82	.16037	.19209	.18552
43	.01187	.01925	.01347	83	.17264	.20225	.20352
44	.01228	.02065	.01408	84	.18342	.21772	.22329
45	.01275	.02155	.01476	85	.19608	.22930	.24491
46	.01327	.02272	.01552	86	.21019	.24123	.26844
47	.01387	.02382	.01637	87	.22643	.25372	.29394
48	.01451	.02493	.01731	88	.24388	.26659	.32143
49	.01521	.02604	.01831	89	.26200	.28000	.35091
50	.01595	.02693	.01935				
51	.01672	.02788	.02043				
52	.01760	.02903	.02156				
53	.01868	.03034	.02276				
54	.01997	.03179	.02408				